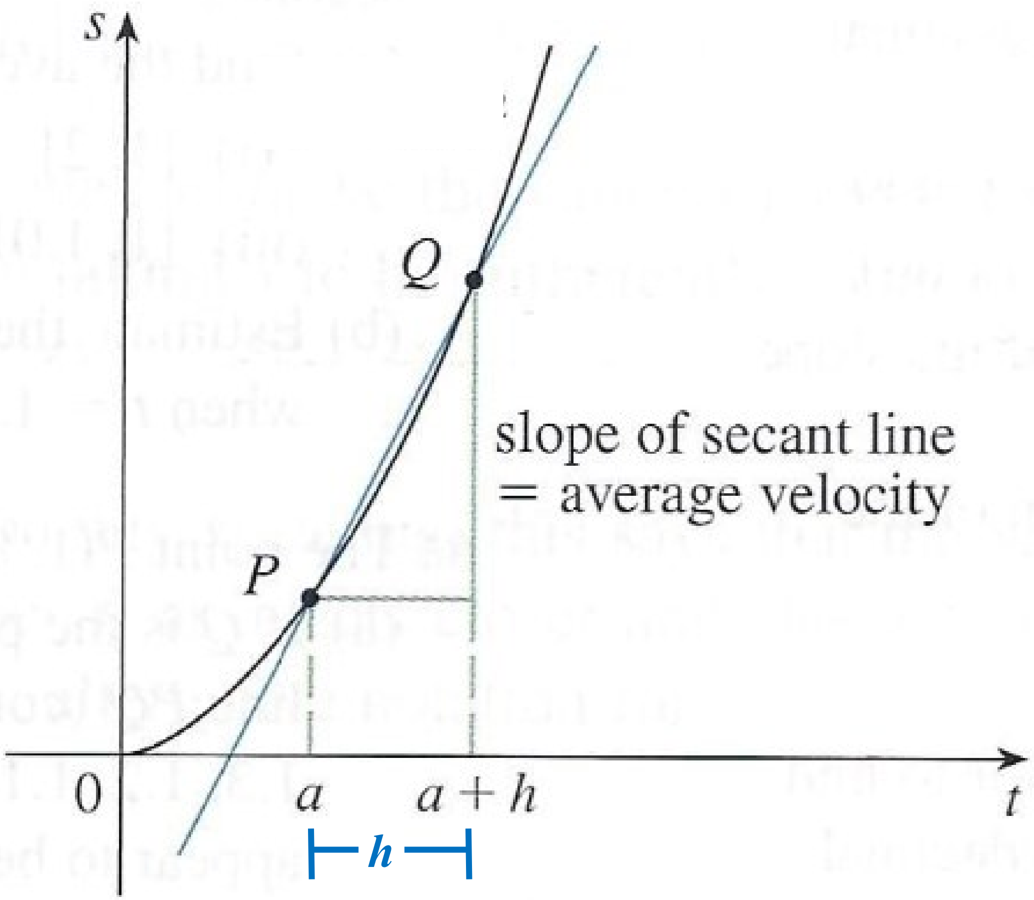
FAR BEYOND

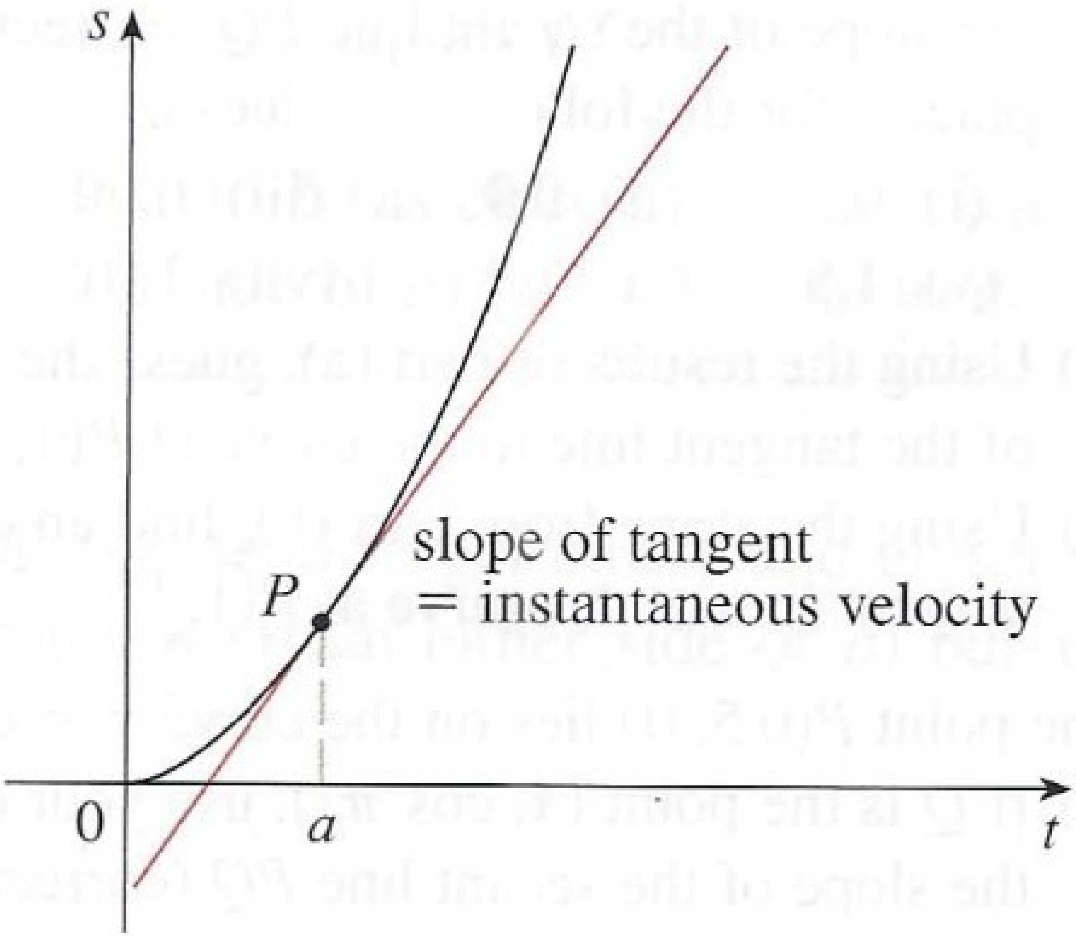
# MAT122

## Definition of the Derivative

Stony Brook University

# Average vs. Instantaneous Velocity



# Velocity Application

Suppose a ball is dropped from upper deck of CN Tower, 450m above the ground.
Find the velocity of the ball after 5 seconds. Use the model $s(t) = 4.9t^2$

average velocity = $m_{\text{sec}}$

$$= \frac{s(5.1) - s(5)}{5.1 - 5}$$

$$= \frac{4.9(5.1)^2 - 4.9(5)^2}{5.1 - 5}$$

$$= \frac{127.449 - 122.5}{.1}$$

$$= 49.49 \ m/s$$

| $a+h$ | $v_{avg}$ |
|---|---|
| 5.1 | 49.49 |
| 5.01 | 49.049 |
| 5.001 | 49.0049 |
| 4.999 | 48.995 |

guess: instantaneous velocity is $49 m/s$

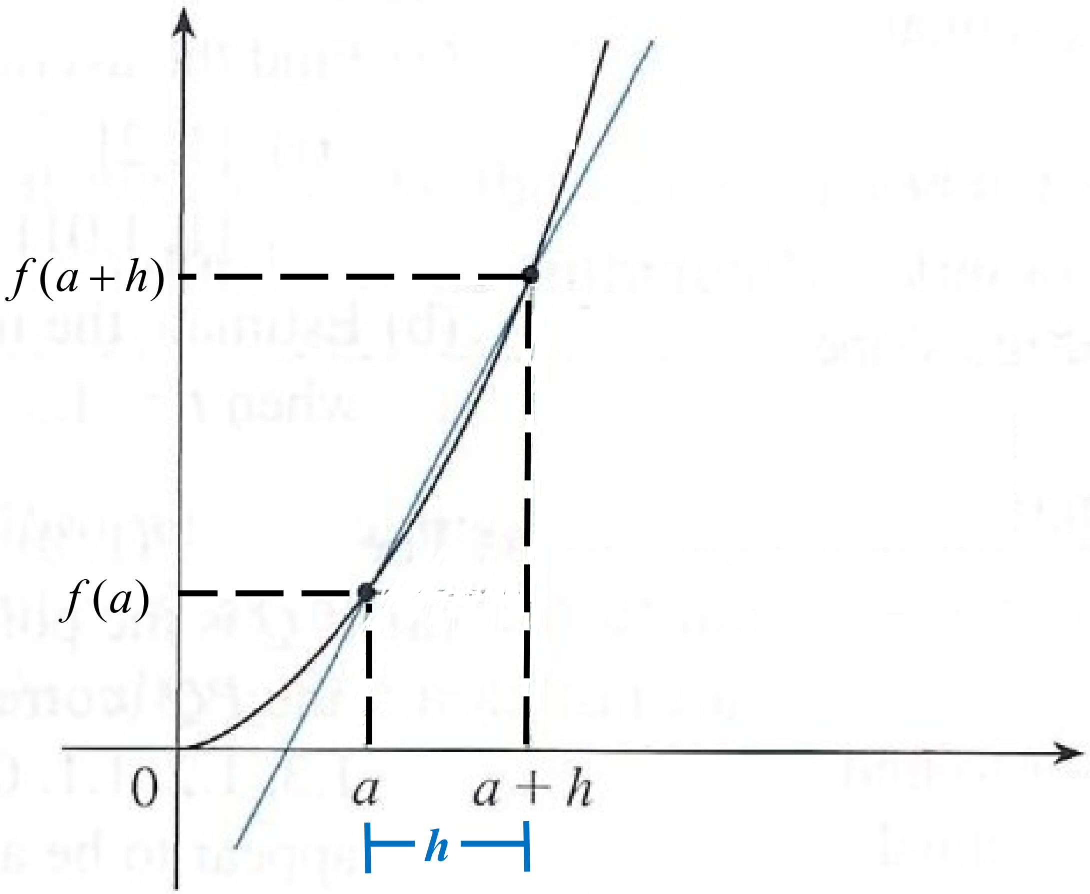
instantaneous velocity = $m_{\tan}$

Technique

Make $h$ **infinitely small**.

Use a **LIMIT** to make distance infinitely small

# Instantaneous Velocity as a Derivative




same as slope of tangent line at $a$

$$v_{inst} = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{a + h - a}$$

difference quotient

(same as rate of change at $a$)

# Definition of the Derivative – Formula w 'a'

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

ex: Find the derivative of $f(x) = x^2 - 8x + 9$ at $x = a$.

$$f(a) = a^2 - 8a + 9$$

$$f(a+h) = (a+h)^2 - 8(a+h) + 9$$

$$= a^2 + 2ah + h^2 - 8a - 8h + 9$$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - (a^2 - 8a + 9)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{a^2} + 2ah + h^2 - \cancel{8a} - 8h + \cancel{9} - \cancel{a^2} + \cancel{8a} - \cancel{9}}{h}$$

$$= \lim_{h\to 0} \frac{2ah + h^2 - 8h}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2a + h - 8)}{\cancel{h}} = \lim_{h\to 0}(2a + h - 8) = 2a + 0 - 8 = \boxed{2a - 8}$$

# Derivative Example with Slope

ex. Find the slope of the tangent line to $y = 4x - x^2$ at $a = 1$ using the definition of the derivative.

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$m = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$f(1+h) = 4(1+h) - (1+h)^2$$

$$= 4 + 4h - (1 + 2h + h^2)$$

$$= 4 + 4h - 1 - 2h - h^2$$

$$= 3 + 2h - h^2$$

$$m = \lim_{h \to 0} \frac{3 + 2h - h^2 - 3}{h}$$

$$= \lim_{h \to 0} \frac{2h - h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2-h)}{h}$$

$$= 2 - 0 = \boxed{2 = m}$$

Follow up question:
Find equation of this tangent line.

use $y - y_1 = m(x - x_1)$ where $(x_1, y_1) = (a, f(a))$

$(1, \quad 3 \quad)$

$$y - 3 = 2\,(x - 1)$$

$$y = 2x - 2 \quad + 3$$

tangent line: $y = 2x + 1$

# General Definition of the Derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

ex: Find the **general** derivative of $f(x) = x^3 - x$

$$f(x+h) = (x+h)^3 - (x+h)$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x} - h - \cancel{x^3} + \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 1)}{\cancel{h}}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1) \quad (3xh \to 0,\ h^2 \to 0)$$

$$f'(x) = \boxed{3x^2 - 1}$$

# General Definition of the Derivative – Example 2

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

ex: Find the general derivative of $f(x) = \sqrt{x}$.

$f(x+h) = \sqrt{x+h}$

multiply by conjugate

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

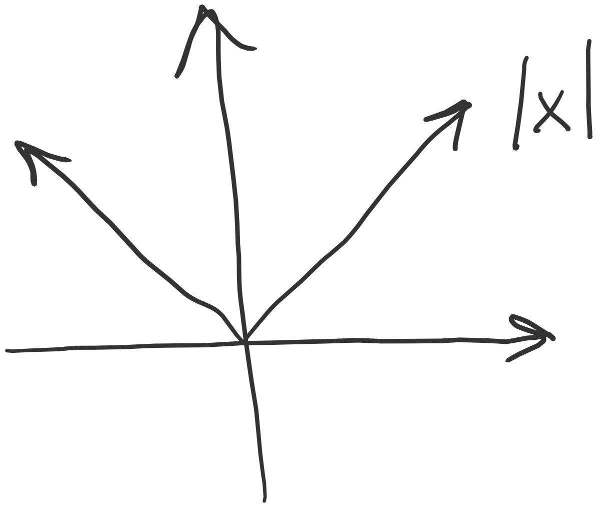
1

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

# Derivative Exceptions

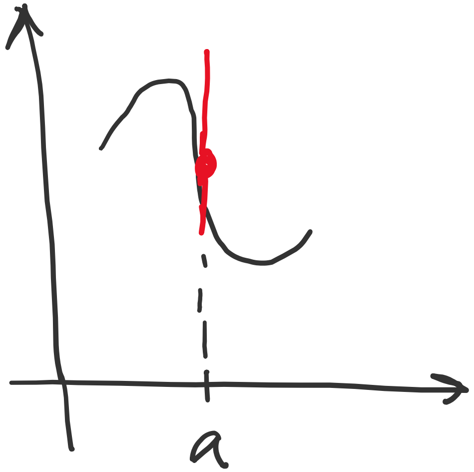



not differentiable at $x = 0$

has a derivative

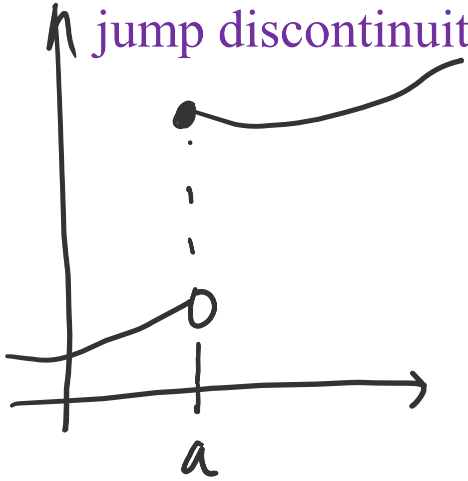
a function is <u>not</u> **differentiable** where there is a:

1. corner
2. discontinuity
3. vertical tangent

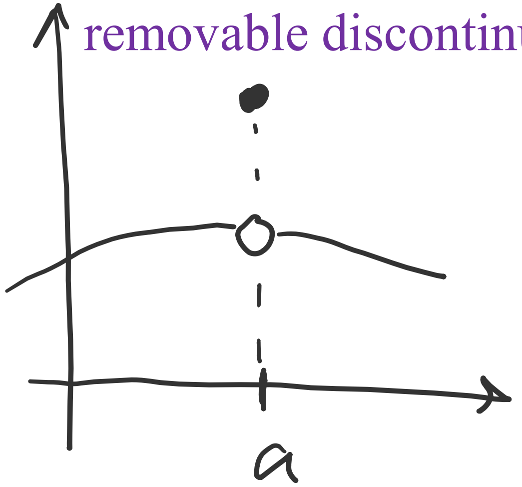




jump discontinuity




removable discontinuity




not differentiable at $x = a$